Note on Inversion of Certain Block-partitioned Matrices

Dedicated to Olga Taussky Todd

Stephen Barnett
School of Mathematics, University of Bradford
Bradford BD7 1DP, *West Yorkshire, England*

Submitted by David H. Carlson

ABSTRACT

Using Kronecker products, a simple expression is derived for the inverse of a certain block-partitioned matrix.

In various applications systems of linear equations

$$Mx = y \tag{1}$$

arise in which M is block-tridiagonal, or "almost" so—for example [5], using a finite-difference approximation to an elliptic partial differential equation. The purpose of this note is to derive a simple expression for the inverse of a particular form of M in (1) which requires inversion of only one or two matrices each having dimension that of the blocks.

The development relies on using the recently introduced [2] $n \times n$ comrade matrix

$$C = \begin{bmatrix} -\beta_1/\alpha_1 & 1/\alpha_1 & 0 & & & \\ \gamma_2/\alpha_2 & -\beta_2/\alpha_2 & 1/\alpha_2 & & \bigcirc & \\ 0 & \gamma_3/\alpha_3 & -\beta_3/\alpha_3 & \ddots & & \\ & \bigcirc & & \ddots & -\beta_{n-1}/\alpha_{n-1} & 1/\alpha_{n-1} \\ -a_n/\alpha_n & -a_{n-1}/\alpha_n & -a_{n-2}/\alpha_n & & (-a_2+\gamma_n)/\alpha_n & (-a_1-\beta_n)/\alpha_n \end{bmatrix}, \tag{2}$$

which has the property that its characteristic polynomial is

$$\tilde{a}(\lambda)=\frac{p_n(\lambda)+a_1p_{n-1}(\lambda)+\cdots+a_np_0(\lambda)}{\alpha_1\alpha_2\cdots\alpha_n} \tag{3}$$

$$=\lambda^n+\tilde{a}_1\lambda^{n-1}+\cdots+\tilde{a}_n, \tag{4}$$

where the polynomials $p_i(\lambda)$ are defined by the relations

$$p_0(\lambda)=1, \qquad p_1(\lambda)=\alpha_1\lambda+\beta_1.$$

$$p_i(\lambda)=(\alpha_i\lambda+\beta_i)p_{i-1}(\lambda)-\gamma_ip_{i-2}(\lambda), \qquad i=2,\ldots,n. \tag{5}$$

The numbers α_i, β_i, γ_i are real and such that $\alpha_i>0$, $\gamma_i\geqslant 0$. (In fact the p_i can be assumed orthogonal.) Clearly (2) is a generalization of the companion matrix, and alternative forms are given in [2].

Consider now the $nq\times nq$ matrix

$$M=I_n\otimes A+C\otimes T \tag{6}$$

$$=\begin{bmatrix} A-(\beta_1/\alpha_1)T & (1/\alpha_1)T & \cdot & \cdot \\ (\gamma_2/\alpha_2)T & A-(\beta_2/\alpha_2)T & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -a_nT/\alpha_n & \cdot & \cdot & A-(a_1+\beta_n)T/\alpha_n \end{bmatrix} \tag{7}$$

where $\otimes$ denotes the Kronecker product, I_n is the unit matrix of order n, and A and T are given $q\times q$ matrices. An expression for the inverse of (6) which relies on the special form of C will be developed below. Of course, the idea of using Kronecker product representations to solve (1) is by no means new. Egerváry [7] considered a case where M is purely block-tridiagonal (all $a_i=0$) and $\alpha_i=1$, $\beta_i=0$, $\gamma_i=1$, so that $\tilde{a}(\lambda)$ in (3) is equal to $S_n(\lambda)$, essentially the Chebyshev polynomial of the second kind. For the particular tridiagonal form C_1 (say) of C in (2) in this case, Egerváry actually shows how to invert the matrix

$$I\otimes(2I-C_1)+(2I-C_1)\otimes I,$$

either by using an explicit form for $(\lambda I-C_1)^{-1}$, or by using the fact that C_1 can be easily diagonalized, since its characteristic vectors and roots [the latter being the zeros of $S_n(\lambda)$] are known. Kershaw [8] has extended

Egerváry's first method by giving inverses for two matrices somewhat more general than $\lambda I - C_1$. Lovass-Nagy and Powers [9] have exploited the second property of C_1 in some recent work. Another procedure for solving (1) when

$$M = I \otimes A + B \otimes I \tag{8}$$

is described in [10] and [11] and relies on diagonalizing A and B, so that their characteristic roots and vectors must be calculated. Yet another way of inverting (8), which holds for arbitrary A and B, is suggested in [3] and [4] and requires determination of the inverse of $\lambda I + B$. It is this last approach which is utilized below. When M in (8) is singular, an expression for its Moore-Penrose inverse has been obtained [6].

This present note gives a procedure for calculating the inverse of a given matrix M in the form (7) for the cases where either T or A is nonsingular, or where $AT = TA$. The approach has two computational advantages. Firstly, the $\tilde{a}_i$ in (4) can be obtained by comparison with (3) once the polynomials $p_i(\lambda)$ have been found by the recursion formula (5). This is less work than computing the coefficients of the characteristic polynomial directly, as in [4]. Secondly, the inverse of M can then be obtained by analogy with the inverse of $\lambda I + C$ along similar lines to those indicated in [4], and the only matrices to be inverted have dimensions $q \times q$. Specifically, it is trivial to verify that for C defined in (2),

$$(\lambda I_n + C)^{-1} = \frac{B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_n}{\det(\lambda I + C)}, \tag{9}$$

where

$$\det(\lambda I + C) = \lambda^n - \tilde{a}_1\lambda^{n-1} + \tilde{a}_2\lambda^{n-2} + \cdots + (-1)^n\tilde{a}_n \tag{10}$$

and

$$B_1 = I_n, \qquad B_i = (-1)^{i-1}\tilde{a}_{i-1}I - CB_{i-1}, \qquad i = 2, \ldots, n.$$

If T is nonsingular, then

$$\begin{aligned} M &= (I_n \otimes T)\left[I_n \otimes (T^{-1}A) + C \otimes I_q\right] \\ &= (I_n \otimes T)N, \end{aligned}$$

say, so that

$$M^{-1}=N^{-1}(I_n\otimes T^{-1}). \tag{11}$$

The blocks in N commute with each other, so by analogy with (9) and (10) it follows that

$$N^{-1}=(I_n\otimes D^{-1})\left[B_1\otimes(T^{-1}A)^{n-1}+B_2\otimes(T^{-1}A)^{n-2}+\cdots+B_{n-1}\otimes I_q\right], \tag{12}$$

where

$$D=(T^{-1}A)^n-\tilde{a}_1(T^{-1}A)^{n-1}+\cdots+(-1)^n\tilde{a}_nI_q, \tag{13}$$

provided D is nonsingular; since by comparison with (10) $\det N=\det D$, it is clear that nonsingularity of D is equivalent to uniqueness of solution of (1) in this case. Thus the inverse of M is obtained from (11) and (12), and the only matrices to be inverted are D in (13) and T, each being $q\times q$.

A similar argument applies when A is nonsingular, by commencing with

$$M=(I_n\otimes A)\left[I_n\otimes I_q+C\otimes(A^{-1}T)\right].$$

Similar reasoning can also be used if $AT=TA$, irrespective of whether T or A is nonsingular, since the blocks in (7) then commute. It is easy to verify that (11) and (12) are replaced by

$$M^{-1}=(I_n\otimes D_1^{-1})$$

$$\left[B_1\otimes A^{n-1}+B_2\otimes A^{n-2}T+\cdots+B_{n-2}\otimes AT^{n-2}+B_{n-1}\otimes T^{n-1}\right],$$

where

$$D_1=A^n-\tilde{a}_1A^{n-1}T+\tilde{a}_2A^{n-2}T^2+\cdots+(-1)^n\tilde{a}_nT^n.$$

Finally, it may be noted that (1) and (6) are equivalent to the matrix equation

$$AX+TXC^{\mathrm{tr}}=Y \tag{14}$$

where X and Y are $q\times n$ matrices whose columns are formed successively from the elements of x and y respectively. The solution X of (14) can then be written down [1] using the expressions for M^{-1}.

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